Invariance properties of random pulses and of other random fields in dispersive media

Weijian Wang and Emil Wolf*
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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This paper is concerned with one-dimensional random fields consisting of randomly distributed identical light pulses which propagate in a linear dispersive medium. It is shown that such fields are necessarily stationary and that their power spectra and their longitudinal coherence properties do not change on propagation. The invariance of the power spectrum and of the coherence properties are shown to apply more generally to any one-dimensional stationary field of any state of coherence, propagating in a linear dispersive medium. Some invariance properties of nondiffracting partially coherent fields in dispersive media and of partially coherent fields in lossless fibers are also discussed.

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I. INTRODUCTION

Propagation of light pulses in dispersive media has been of interest for a long time. The first important contribution to this subject was made by Brillouin and Sommerfeld who, in two well-known papers published in 1914, clarified some basic questions relating to their velocity of propagation [1–3]. Since the development of lasers in the 1960s, light pulses of shorter and shorter duration have been produced for a variety of applications. In spite of the numerous investigations concerning pulses, some rather basic questions relating to their spectral and coherence properties are still rather poorly understood. It is the aim of this paper to clarify some of these questions.

This work was motivated by some unexpected results found in interference experiments involving electron and neutron beams [4–11]. Figure 1 shows a schematic diagram of an experiment on longitudinal coherence length of neutrons carried out by Kaiser, Werner, and George [5]. A neutron beam was divided into two beams which were then superposed. The visibility of interference fringes was measured as a function of the thickness D of the potential barrier, or, equivalently, as a function of the relative delay between the two beams traveling through the two paths I and II. Because the neutron wave packets in the original beam are statistically independent of each other, the observed interference fringes are generated by interference of each wave packet with only itself. Hence, if the relative delay introduced between the two paths is longer than the duration of the neutron wave packets, no interference fringes should be found. This suggests that the longitudinal coherence time (or, equivalently, the coherence length) of the neutron wave packets is equal to their duration (or length). However, according to quantum mechanics, the neutron wave packets spread as they propagate [12], i.e., their duration and length increase on propagation. One might, therefore, assume that the coherence time and the coherence length of the neutron wave packets will also increase as they propagate. However, the experiment showed that this is no so. It was found from the measurements of the visibility of interference fringes that the longitudinal coherence length of the neutron wave packets is equal to their effective initial length, *not* to their (increased) length after propagation [5]. This implies that the coherence length remains unchanged, even though the wave packets spread on propagation.

The first theoretical explanation of this result was provided by Klein, Opat, and Hamilton [13]. They also suggested that, because of the well-known analogy between the propagation of free particles in quantum mechanics and that of classical light pulses in (linear) dispersive media, similar results should hold for light pulses propagating in dispersive media. However, there is an essential difference between the two cases, not mentioned in the paper by Klein, Opat, and Hamilton [13], namely, that causality demands that a dispersive medium is necessarily also absorbing. Nevertheless, under certain conditions, i.e., when all the resonance frequencies of the medium are

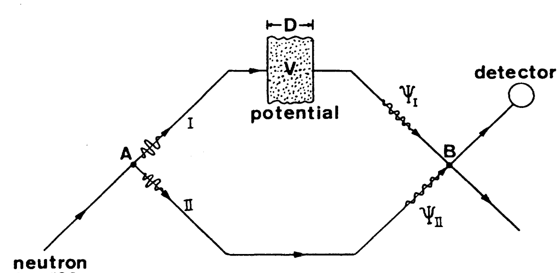

FIG. 1. Schematic diagram of an interference experiment with neutron beams. (After Kaiser, Werner, and George [5].)

*Also at the Institute of Optics, University of Rochester, Rochester, NY 14627.

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far from the effective frequencies of the light field, absorption may be neglected and the medium then behaves as a purely dispersive medium.

In this paper we first investigate one-dimensional propagation, in a purely dispersive linear medium, of field generated by sources which emit identical light pulses at random instants of time, with a fixed average emission rate, and we show that such fields are necessarily stationary and that their spectral and coherence properties do not change on propagation. We then establish some other invariance properties of partially coherent nondiffracting fields in dispersive media and of partially coherent fields in lossless fibers.

II. PROPAGATION OF RANDOM LIGHT PULSES IN DISPERSIVE MEDIA

Consider a source that emits identical light pulses which then propagate along some specific direction, $+z$ say, in a linear dispersive medium. We assume that the electric vector of each pulse is linearly polarized in a direction specified by a unit vector $\mathbf{e}_0$, which is perpendicular to the z direction. We may represent the initial shape of the pulse by the expression $\Lambda(t)\mathbf{e}_0$, where t denotes the time and $\Lambda(t)$ is localized in a short time interval $(-\delta t/2, \delta t/2)$. Suppose that the source is located at $z=0$ and that the pulses are emitted at random instants of time. Then the resulting fluctuating field may be represented by an ensemble $\{V(z,t)\mathbf{e}_0\}$, each realization of which at $z=0$ may be expressed in the form

$$V(0,t)=\sum_j \Lambda(t-t_j) , \tag{2.1}$$

where the t_j's are randomly distributed.

Consider first the pulse field in a finite time interval $[-T/2, T/2]$. Assuming that the probability of emission of a pulse in a short time interval $[t, t+\Delta t]$ is statistically independent of earlier emissions and that it is proportional to Δt, it can be shown that the probability that exactly N pulses are emitted in the interval $[-T/2, T/2]$ is given by the Poisson distribution [14]

$$p(N)=\frac{\bar{N}^N}{N!}e^{-\bar{N}} . \tag{2.2}$$

Here $\bar{N}$ is the average of N and is proportional to T:

$$\bar{N}=\eta T , \tag{2.3}$$

η being the average pulse emission rate, assumed to be constant.

If $V_N(z,t)\mathbf{e}_0$ represents a realization of the subensemble $\{V_N(z,t)\mathbf{e}_0\}$ formed by all the realizations of the ensemble $\{V(z,t)\mathbf{e}_0\}$ that contain exactly N pulses, then at $z=0$

$$V_N(0,t)=\sum_{j=1}^{N} \Lambda(t-t_j) . \tag{2.4}$$

We may take $V_N(z,t)$ and $\Lambda(t)$ to be the complex analytical signal representation [15,16] of the corresponding real quantities. Then their Fourier representations contain only non-negative frequencies:

$$V_N(z,t)=\int_0^\infty \tilde{V}_N(z,\omega)e^{-i\omega t}d\omega , \tag{2.5}$$

$$\Lambda(t)=\int_0^\infty \tilde{\Lambda}(\omega)e^{-i\omega t}d\omega . \tag{2.6}$$

Evidently,

$$\tilde{V}_N(z,\omega)=\frac{1}{2\pi}\int_{-T/2}^{T/2} V_N(z,t)e^{i\omega t}dt , \tag{2.7}$$

$$\tilde{\Lambda}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \Lambda(t)e^{i\omega t}dt . \tag{2.8}$$

Each realization $\tilde{V}_N(z,\omega)$ satisfies the Helmholtz equation

$$\left[\frac{\partial^2}{\partial z^2}+k^2\right]\tilde{V}_N(z,\omega)=0 , \tag{2.9}$$

where

$$k\equiv k(\omega)=n(\omega)k_0 \tag{2.10}$$

is the wave number in the dispersive medium, $k_0=\omega/c$ is the free-space wave number associated with the frequency ω (c being the speed of light in vacuum), and $n(\omega)$ is the refractive index of the medium, assumed to be real. In order that $n(\omega)$ is real, the effective frequencies of $\tilde{\Lambda}(\omega)$ must be far from any resonance of the medium.

One can readily show that the general solution to Eq. (2.9) for propagation along the positive z direction is

$$\tilde{V}_N(z,\omega)=A_N(\omega)e^{ik(\omega)z} , \tag{2.11}$$

where the $A_N(\omega)$'s are arbitrary. To determine the $A_N(\omega)$'s for the present problem, we first substitute from Eq. (2.11) into (2.5) and we find that

$$V_N(z,t)=\int_0^\infty A_N(\omega)e^{i[k(\omega)z-\omega t]}d\omega . \tag{2.12}$$

On setting $z=0$ in this expression and making use of Eq. (2.4), we find that

$$\sum_{j=1}^{N} \Lambda(t-t_j)=\int_0^\infty A_N(\omega)e^{-i\omega t}d\omega . \tag{2.13}$$

Taking the Fourier transform of the above equation and using Eq. (2.8), we obtain for $A_N(\omega)$ the expression

$$A_N(\omega)=\sum_{j=1}^{N} \tilde{\Lambda}(\omega)e^{i\omega t_j} . \tag{2.14}$$

We have assumed here that $\delta t \ll T$, i.e., that the duration of $\Lambda(t)$ is much shorter than the time interval under consideration, and that all pulses are contained within the time interval $[-T/2, T/2]$. On substituting from Eq. (2.14) into (2.12), we find that

$$V_N(z,t)=\int_0^\infty \sum_{j=1}^{N} \tilde{\Lambda}(\omega)e^{i\omega t_j}e^{i[k(\omega)z-\omega t]}d\omega . \tag{2.15}$$

We have so far considered a single realization of the ensemble $\{V_N(z,t)\mathbf{e}_0\}$. Suppose now that the emission times t_j's of the pulses are statistically independent, i.e., that the joint probability density $p(t_1, t_2, \ldots, t_N)$ that N pulses are emitted at the times $t_1, t_2, \ldots, t_N$ factorizes, viz.,

$$p(t_1,t_2,\ldots,t_N)=p(t_1)p(t_2)\cdots p(t_N)\;, \tag{2.16}$$

where $p(t)$ is the probability density that a pulse is emitted at the time t. We also assume that $p(t)$ is uniform in the time interval $(-T/2,T/2)$, i.e., that

$$p(t)dt=dt/T\;. \tag{2.17}$$

On using Eqs. (2.15)–(2.17), the ensemble average (denoted by angular brackets) of $V_N(z,t)$ is found to be given by the formula

$$\langle V_N(z,t)\rangle=\int_{-T/2}^{T/2}\frac{dt_1}{T}\cdots\int_{-T/2}^{T/2}\frac{dt_N}{T}\int_0^\infty\sum_{j=1}^{N}\tilde{\Lambda}(\omega)e^{i\omega t_j}e^{ik(\omega)z-i\omega t}d\omega$$
$$=\frac{N}{T}\int_0^\infty\tilde{\Lambda}(\omega)\left[\frac{\sin(\omega T/2)}{\omega/2}\right]e^{ik(\omega)z-i\omega t}d\omega\;. \tag{2.18}$$

In a similar way, one finds that

$$\langle V_N^*(z,t)V_N(z',t')\rangle=\int_0^\infty\int_0^\infty d\omega\,d\omega'\tilde{\Lambda}^*(\omega)\tilde{\Lambda}(\omega')e^{-ik(\omega)z+i\omega t}e^{ik(\omega')z'-i\omega' t'}\sum_{j,l=1}^{N}\langle e^{-i\omega t_j+i\omega' t_l}\rangle\;, \tag{2.19}$$

where

$$\sum_{j,l=1}^{N}\langle e^{-i\omega t_j+i\omega' t_l}\rangle=\sum_{j=1}^{N}\langle e^{i(\omega'-\omega)t_j}\rangle+\sum_{j\neq l}^{N}\langle e^{-i\omega t_j}\rangle\langle e^{i\omega' t_l}\rangle$$
$$=\frac{N}{T}\frac{\sin[(\omega'-\omega)T/2]}{(\omega'-\omega)/2}+\frac{N(N-1)}{T^2}\frac{\sin(\omega T/2)}{\omega/2}\frac{\sin(\omega' T/2)}{\omega'/2}\;. \tag{2.20}$$

It is to be noted that these averages have been taken over the subensemble $\{V_N(z,t)\mathbf{e}_0\}$.

Next let us consider the complete ensemble $\{V(z,t)\mathbf{e}_0\}$. The average of $V(z,t)$ is given by the expression

$$\langle V(z,t)\rangle=\sum_{N=0}^{\infty}p(N)\langle V_N(z,t)\rangle=\eta\int_0^\infty\tilde{\Lambda}(\omega)\left[\frac{\sin(\omega T/2)}{\omega/2}\right]e^{ik(\omega)z-i\omega t}d\omega\;, \tag{2.21}$$

and the cross-correlation function of $V(z,t)$ at two space-time points is given by the formula

$$\Gamma(z,t;z',t')\equiv\langle V^*(z,t)V(z',t')\rangle=\sum_{N=0}^{\infty}p(N)\langle V_N^*(z,t)V_N(z',t')\rangle$$
$$=\eta\int_0^\infty\int_0^\infty d\omega\,d\omega'\frac{\sin[(\omega'-\omega)T/2]}{(\omega'-\omega)/2}\tilde{\Lambda}^*(\omega)\tilde{\Lambda}(\omega')e^{-ik(\omega)z+i\omega t}e^{ik(\omega')z'-i\omega' t'}$$
$$+\langle V^*(z,t)\rangle\langle V(z',t')\rangle\;, \tag{2.22}$$

where Eqs. (2.2) and (2.18)–(2.20) have been used.

Let us now proceed to the limit $T\to\infty$. In this limit,

$$\frac{\sin(\omega T/2)}{\omega/2}\to4\pi\delta^{(e)}(\omega)\;, \tag{2.23}$$

$$\frac{\sin[(\omega'-\omega)T/2]}{(\omega'-\omega)/2}\to2\pi\delta(\omega'-\omega)\;. \tag{2.24}$$

The function $\delta^{(e)}(\omega)$ in the formula (2.23) is the even half-δ function, defined by the two properties

$$\delta^{(e)}(\omega)=0\quad\text{when }\omega\neq0 \tag{2.25}$$

and

$$\int_0^\epsilon\delta^{(e)}(\omega)d\omega=\tfrac{1}{2}\quad\text{when }\epsilon>0\;. \tag{2.26}$$

On substituting from expressions (2.23) and (2.24) into Eqs. (2.21) and (2.22), we find that

$$\langle V(z,t)\rangle=2\pi\eta\tilde{\Lambda}(0)\;, \tag{2.27}$$

$$\Gamma(z,t;z',t')=2\pi\eta\int_0^\infty|\tilde{\Lambda}(\omega)|^2e^{ik(\omega)(z'-z)-i\omega(t'-t)}d\omega$$
$$+\langle V^*(z,t)\rangle\langle V(z',t')\rangle\;. \tag{2.28}$$

It is seen that $\langle V(z,t)\rangle$ is independent of t and $\Gamma(z,t;z',t')$ depends on t and t' only through the difference $\tau=(t'-t)$. These properties imply that the field is statistically stationary, at least in the wide sense [17,18]. Consequently, we can write $\Gamma(z,z',t'-t)$ in place of $\Gamma(z;t;z',t')$. The function $\Gamma(z,z',t'-t)$ is the one-dimensional form of the well-known mutual coherence function of optical coherence theory [16,19]. On using Eq. (2.8), the expressions (2.27) and (2.28) become

$$\langle V(z,t)\rangle = \eta\int_{-\infty}^{\infty}\Lambda(t)dt\ , \tag{2.29}$$

$$\Gamma(z,z',\tau) = \eta C_F(z,z',\tau) + \left[\eta\int_{-\infty}^{\infty}F(0,t)dt\right]^2\ , \tag{2.30}$$

where

$$C_F(z,z',\tau) \equiv \int_{-\infty}^{\infty}F^*(z,t)F(z',t+\tau)dt\ , \tag{2.31}$$

with

$$F(z,t) = \int_0^{\infty}\tilde{\Lambda}(\omega)e^{i[k(\omega)z-\omega t]}d\omega\ . \tag{2.32}$$

It is clear that

$$F(0,t) = \Lambda(t) \tag{2.33}$$

and that $F(z,t)$ represents the shape of the pulse $\Lambda(t)$ after the pulse has propagated in the dispersive medium to z from $z=0$. The function $C_F(z,z',\tau)$ represents the (deterministic) correlation of $F(z,t)$ and $F(z',t+\tau)$ and we will, therefore, call it the *single-pulse correlation function*. Expressions (2.29) and (2.30) are analogous to the well-known expressions encountered in the theory of the shot noise, known as Campbell's theorem [20–22].

Equation (2.30) expresses $\Gamma(z,z',\tau)$ in terms of the single-pulse correlation function $C_F(z,z',\tau)$. From Eq. (2.32), it follows that as the pulse propagates in the (linear) dispersive medium, it becomes broader, i.e., the duration of $F(z,t)$ increases with z, in general. Hence one might expect that the single-pulse correlation function $C_F(z,z',\tau)$ of $F(z,t)$ and $F(z',t+\tau)$ as a function of τ becomes broader when z and z' increase [with $(z'-z)$ fixed]. This, however, is not so, because as can be shown from Eqs. (2.31) and (2.32)

$$C_F(z,z',\tau) = C_F(0,z'-z,\tau)\ . \tag{2.34}$$

This result implies that the correlation between $F(z,t)$ and $F(z',t+\tau)$ does not change on propagation [with $(z'-z)$ fixed]. This fact becomes perhaps clearer if one considers the special case when $z'=z$. It follows from Eq. (2.34) that $C_F(z,z,\tau)=C_F(0,0,\tau)$, or on using Eqs. (2.31) and (2.33)

$$\int_{-\infty}^{\infty}F^*(z,t)F(z,t+\tau)dt = \int_{-\infty}^{\infty}\Lambda^*(t)\Lambda(t+\tau)dt\ , \tag{2.35}$$

showing that the quantity on the left-hand side does not depend on z. Hence, if the relative delay τ is greater than the duration δt of the initial pulse $\Lambda(t)$, there will be no correlation, even though $F^*(z,t)$ and $F(z,t+\tau)$ may overlap at z (Fig. 2), because the right-hand side of the above equation vanishes. Now when $z'=z$, Eq. (2.30) reduces to

$$\begin{aligned}\Gamma(z,z,\tau) &= \eta C_F(0,0,\tau) + \left[\eta\int_{-\infty}^{\infty}\Lambda(t)dt\right]^2 \\ &= \Gamma(0,0,\tau)\ ,\end{aligned} \tag{2.36}$$

showing that the self-coherence function $\Gamma(z,z,\tau)$ is independent of z.

It is also evident from Eqs. (2.30) and (2.34) that

$$\Gamma(z,z',\tau) = \Gamma(0,z'-z,\tau)\ , \tag{2.37}$$

which, together with Eq. (2.29), implies that the field is

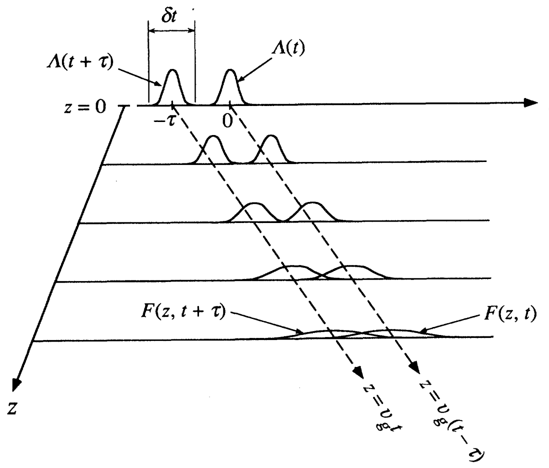

FIG. 2. Evolution, in a dispersive medium, of the envelopes of $F(z,t)$ and $F(z,t+\tau)$. Because of the spreading of the pulse on propagation, $F(z,t)$ and $F(z,t+\tau)$ eventually overlap, although they were well separated before propagation. v_g denotes the group velocity in the medium.

statistically homogeneous [23]. Therefore the second-order coherence characteristics (more specifically, the coherence time and the coherence length) of the pulse field do not change on propagation. Moreover, the power spectrum which, by the Wiener-Khintchine theorem [24,25], is equal to the Fourier transform of the self-coherence function $\Gamma(z,z,\tau)$, is found, on using Eq. (2.36), to be given by the expression

$$S(z,\omega) = 2\pi\eta|\tilde{\Lambda}(\omega)|^2 + 16\pi^3\eta^2\tilde{\Lambda}^2(0)\delta^{(e)}(\omega)\ . \tag{2.38}$$

This formula shows that the power spectrum of the field is (i) invariant on propagation, even though the medium is (purely) dispersive and (ii) proportional to the square of the modulus of the Fourier spectrum of a single pulse, except for an additive dc component.

We have demonstrated in this section the invariance, on propagation, of the power spectrum and of the mutual coherence function of a one-dimensional pulse field. Actually, the invariance is more general: it holds for any one-dimensional wide-sense stationary field of any state of coherence, propagating in any linear purely dispersive medium. In fact, it has been shown [26] that the mutual coherence function for a general one-dimensional wide-sense stationary field is given by the expression

$$\Gamma(z,z',\tau) = \int_0^{\infty}S_0(\omega)e^{i[k(\omega)(z'-z)-\omega\tau]}d\omega\ , \tag{2.39}$$

where $S_0(\omega)$ is the power spectrum of the field at the origin $z=0$. Since $\Gamma(z,z',\tau)$ is seen to depend on z and z' only through the difference $(z'-z)$, such a field is necessarily statistically homogeneous [27], even though the medium is (purely) dispersive. Consequently, the (longitudinal) second-order coherence properties of the field are invariant on propagation. It is also clear from Eq. (2.39), on taking $z'=z$ and using the Wiener-Khintchine theorem, that

$$S(z,\omega)=S_0(\omega)\ , \tag{2.40}$$

implying that the power spectrum $S(z,\omega)$ at any point z, just as in the case of the pulse field, is independent of z, i.e., it is invariant on propagation.

We have so far considered propagation in one dimension only. It is natural to ask whether the invariance of the power spectrum and of the coherence properties of a partially coherent field holds also in three dimension. The answer is negative, in general, since as is well known coherence properties of partially coherent light change on propagation, a fact which follows, for example, from the well-known van Cittert–Zernike theorem [28]. It has also been demonstrated that, in general, the power spectrum changes on propagation, even in free space [29,30]. This fact is a consequence of rather subtle interference effects in fluctuating wave fields. Roughly speaking, as a result of interference, the spectral component of a particular frequency may have a maximum at some point in space; on the other hand, the spectral component of a different frequency may have a minimum there. The situation may be quite different at other points and, consequently, the spectrum will, in general, be different at different points in space. Nevertheless, light fields whose power spectra and coherence properties do not change on propagation are realizable under certain circumstances. We will now consider some fields of this kind.

III. INVARIANCE PROPERTIES OF NONDIFFRACTING PARTIALLY COHERENT FIELDS IN DISPERSIVE MEDIA

Consider a nondiffracting partially coherent field, propagating along the positive z direction in a linear dispersive medium whose refractive index $n(\omega)$ in the frequency region of interest may be assumed to be real. The cross-spectral density of the field can be expressed in the form [31]

$$W(\mathbf{r},\mathbf{r}',\omega)=\int_0^{k(\omega)}df\, e^{i(z'-z)\sqrt{k^2(\omega)-f^2}} \times\int\int_0^{2\pi}d\phi\, d\phi' f^2 A(f,\phi,\phi',\omega) \times e^{if(x'\cos\phi'+y'\sin\phi')} \times e^{-if(x\cos\phi+y\sin\phi)}\ , \tag{3.1}$$

where $k(\omega)$ is again given by Eq. (2.10), $\mathbf{r}=(x,y,z)$, $\mathbf{r}'=(x',y',z')$, and the function $A(f,\phi,\phi',\omega)$ characterizes the angular correlation of the field at the frequency ω, at two azimuthal angles ϕ and ϕ', for a fixed f [32]. The mutual coherence function is given by the Fourier transform of the cross-spectral density $W(\mathbf{r},\mathbf{r}',\omega)$, viz.,

$$\Gamma(\mathbf{r},\mathbf{r}',\tau)=\int_0^{\infty}d\omega\, e^{-i\omega\tau}\int_0^{k(\omega)}df\, e^{i(z'-z)\sqrt{k^2(\omega)-f^2}}\int_0^{2\pi}\int_0^{2\pi}d\phi\, d\phi' f^2 A(f,\phi,\phi',\omega)e^{if(x'\cos\phi'+y'\sin\phi')}e^{-if(x\cos\phi+y\sin\phi)}\ . \tag{3.2}$$

We see that the mutual coherence function $\Gamma(\mathbf{r},\mathbf{r}',\tau)$ depends on z and z' only through the difference $(z'-z)$, which implies that the field is statically homogeneous along the positive z direction. Consequently, as we will now show, the power spectrum and the second-order coherence properties of the field will not change as the field propagates.

An expression for the power spectrum $S(\mathbf{r},\omega)$ of the field is obtained at once by setting $\mathbf{r}'=\mathbf{r}$ in Eq. (3.1):

$$S(\mathbf{r},\omega)=\int_0^{k(\omega)}df\, f^2\int\int_0^{2\pi}d\phi\, d\phi' A(f,\phi,\phi',\omega)e^{if[x(\cos\phi'-\cos\phi)+y(\sin\phi'-\sin\phi)]}\ . \tag{3.3}$$

Since the right-hand side is independent of z, the power spectrum is invariant along the positive z direction.

Next let us consider the longitudinal coherence properties of the field. The mutual coherence function of the field at two points (x_0,y_0,z) and $(x_0,y_0,z+\xi)$ located on a line parallel to the z axis is, according to Eq. (3.2), given by the expression

$$\Gamma(x_0,y_0,z;x_0,y_0,z+\xi;\tau) =\int_0^{\infty}d\omega\, e^{-i\omega\tau}\int_0^{k(\omega)}df\, e^{i\xi\sqrt{k^2(\omega)-f^2}}\int_0^{2\pi}\int_0^{2\pi}d\phi\, d\phi' A(f,\phi,\phi',\omega)e^{if[x_0(\cos\phi'-\cos\phi)+y_0(\sin\phi'-\sin\phi)]}\ , \tag{3.4}$$

which is seen to be independent of z. Hence the longitudinal second-order coherence properties do not change along the positive z direction.

Finally let us consider the transverse coherence properties of the field. For two points in a plane, $z=z_0>0$ say, which is perpendicular to the z axis, Eq. (3.2) takes the form

$$\Gamma(x,y,z_0;x',y',z_0;\tau)=\int_0^{\infty}d\omega\, e^{-i\omega\tau}\int_0^{k(\omega)}df\, f^2\int_0^{2\pi}\int_0^{2\pi}d\phi\, d\phi' A(f,\phi,\phi',\omega)e^{if(x'\cos\phi'+y'\sin\phi')}e^{-if(x\cos\phi+y\sin\phi)}\ , \tag{3.5}$$

showing that $\Gamma(x,y,z_0;x',y',z_0;\tau)$ is independent of z_0. This result implies that the transverse second-order coherence properties of the field (i.e., the second-order correlations of the field in any plane $z=z_0=\text{const}$) do not change on propagation.

IV. PROPAGATION IN A FIBER

Let us now consider propagation of partially coherent light in a fiber whose axis is along the z direction. We assume that the fiber is lossless, and we denote by $n(\rho,\omega)$

$[\boldsymbol{\rho}=(x,y)]$ its (real) refractive index.

Usually a fluctuating field is represented by an ensemble $\{V(\mathbf{r},t)\}$, each realization $V(\mathbf{r},t)$ of which may be regarded as a Cartesian component of the transverse electric field vector. For our purpose it is, however, more convenient to carry out the analysis in the space-frequency domain. The cross-spectral density of the field may then be expressed in the form

$$W(\mathbf{r},\mathbf{r}',\omega)=\langle U^*(\mathbf{r},\omega)U(\mathbf{r}',\omega)\rangle_\omega , \qquad (4.1)$$

where the $U(\mathbf{r},\omega)$'s are members of a suitably chosen statistical ensemble $\{U(\mathbf{r},\omega)e^{-i\omega t}\}$ of monochromatic realizations [33–36] of a Cartesian component of the transverse electric field vector. The angular brackets, with a subscript ω, denote the average taken over this ensemble. We have assumed here that the field is stationary, at least in the wide sense. We further assume that the fiber is semi-infinite and that the field propagates from the input face $z=0$ into the region $z>0$ containing the fiber. Each realization $U(\mathbf{r},\omega)$ may then be expressed in terms of the modes of the fiber as [37]

$$U(\mathbf{r},\omega)=\sum_m a_m(\omega)u_m(\boldsymbol{\rho},\omega)e^{i\beta_m(\omega)z} , \qquad (4.2)$$

where the $u_m(\boldsymbol{\rho},\omega)$'s are the orthonormal transverse modes of the field in the fiber, the $\beta_m(\omega)$'s are the propagation constants of the modes, the $a_m(\omega)$'s are random variables, and $\mathbf{r}=(\boldsymbol{\rho},z)$. The single summation index m is to be understood as representing collectively all the indices which label a particular mode. The summation symbol may denote either summation over the discrete guided (propagation) modes or integration over a possible continuum of radiation modes.

We will consider only the guided modes. The summation in Eq. (4.2) is then over such modes only and the propagation constants are real and non-negative:

$$\beta_m(\omega)\geq 0 . \qquad (4.3)$$

On substituting from Eq. (4.2) into (4.1), we obtain the following expression for the cross-spectral density of the field in the fiber:

$$W(\mathbf{r},\mathbf{r}',\omega)=\sum_m\sum_{m'}\langle a_m^*(\omega)a_{m'}(\omega)\rangle_\omega u_m^*(\boldsymbol{\rho},\omega)u_{m'}(\boldsymbol{\rho}',\omega) \times e^{-i\beta_m(\omega)z+i\beta_{m'}(\omega)z'} . \qquad (4.4)$$

On taking the Fourier transform of this expression, we find that the mutual coherence function of the field is given by the formula

$$\Gamma(\mathbf{r},\mathbf{r}',\tau)=\int_0^\infty d\omega\sum_m\sum_{m'}\langle a_m^*(\omega)a_{m'}(\omega)\rangle_\omega \times u_m^*(\boldsymbol{\rho},\omega)u_{m'}(\boldsymbol{\rho}',\omega) \times e^{i[-\beta_m(\omega)z+\beta_{m'}(\omega)z'-\omega\tau]} . \qquad (4.5)$$

We see that the mutual coherence function $\Gamma(\mathbf{r},\mathbf{r}',\tau)$ depends, in general, on both z and z', not just on their difference. This implies that a partially coherent field in the fiber does not, in general, possess the invariance properties of one-dimensional fields and of nondiffracting fields which are discussed earlier. However, under certain conditions, a partially coherent light propagating in the fiber will be statistically homogeneous along the axis of the fiber, as we will now show.

Single-mode fibers

Suppose that all the effective frequencies of the light are below the second-lowest cutoff frequency of the fiber. Then only the fundamental mode of the fiber will be excited and Eq. (4.5) reduces to

$$\Gamma(\mathbf{r},\mathbf{r}',\tau)=\int_0^\infty W_0(\boldsymbol{\rho},\boldsymbol{\rho}',\omega)e^{i[\beta_0(\omega)(z'-z)-\omega\tau]}d\omega , \qquad (4.6)$$

where

$$W_0(\boldsymbol{\rho},\boldsymbol{\rho}',\omega)=A_0(\omega)u_0^*(\boldsymbol{\rho},\omega)u_0(\boldsymbol{\rho}',\omega) , \qquad (4.7)$$

with

$$A_0(\omega)=\langle|a_0(\omega)|^2\rangle_\omega . \qquad (4.8)$$

The function $u_0(\boldsymbol{\rho},\omega)$ represents the fundamental mode, with $\beta_0(\omega)$ being its propagation constant. From Eqs. (4.6) and (4.7), it follows that the cross-spectral density of the field in the fiber is given by the expression

$$W(\mathbf{r},\mathbf{r}',\omega)=A_0(\omega)[u_0^*(\boldsymbol{\rho},\omega)e^{-i\beta_0(\omega)z}] \times[u_0(\boldsymbol{\rho}',\omega)e^{i\beta_0(\omega)z'}] . \qquad (4.9)$$

Because $W(\mathbf{r},\mathbf{r}',\omega)$ factorizes with respect to $\mathbf{r}$ and $\mathbf{r}'$, the field in the fiber is now completely spatially coherent at each frequency [38]. However, it is not necessarily completely coherent in the space-time domain (cf. Ref. [39]).

We see from the expression (4.6) that $\Gamma(\mathbf{r},\mathbf{r}',\tau)$ depends on z and z' only through the difference $(z'-z)$. Hence the field in the single-mode fiber is statistically homogeneous along the axis of the fiber. This result can readily be shown to imply that the spectral and the second-order longitudinal and transverse coherence properties of the field are invariant on propagation along the fiber.

Incoherently excited multimode fibers

Let us now consider the situation when the transverse modes in the fiber are incoherently excited, i.e, when the different modes are uncorrelated. We then have [40]

$$\langle a_m^*(\omega)a_{m'}(\omega)\rangle_\omega=A_m(\omega)\delta_{mm'} , \qquad (4.10)$$

where $\delta_{mm'}$ is the Kronecker delta symbol. On substituting from Eq. (4.10) into Eq. (4.5), we obtain, for the mutual coherence function, the expression

$$\Gamma(\mathbf{r},\mathbf{r}',\tau)=\int_0^\infty d\omega\sum_m A_m(\omega)u_m^*(\boldsymbol{\rho},\omega)u_m(\boldsymbol{\rho}',\omega) \times e^{i[\beta_m(\omega)(z'-z)-\omega\tau]} . \qquad (4.11)$$

We see that $\Gamma(\mathbf{r},\mathbf{r}',\tau)$ again depends on z and z' only through the difference $(z'-z)$. Hence the field in an incoherently excited multimode fiber is statistically homogeneous along the axis of the fiber (the z axis). This result can readily be shown to imply that the power spectrum and the second-order longitudinal and transverse coherence properties of the field do not change on propagation along the fiber.

V. CONCLUSIONS

We have shown that a one-dimensional field consisting of random, independently emitted, identical pulses, propagating in a purely dispersive medium, is statistically stationary, at least in the wide sense, and is statistically homogeneous; and that the power spectrum and the longitudinal (second-order) coherence properties of the field do not change on propagation. Its mutual coherence function was found to be expressible in terms of the single-pulse correlation function. Except for a dc component, the power spectrum of the field was found to be proportional to the square of the modulus of the Fourier spectrum of a single pulse.

We have also considered the propagation of the nondiffracting partially coherent field in purely dispersive media. We have shown that, the field is necessarily statistically homogeneous along its direction of propagation and that its power spectrum and its longitudinal and transverse (second-order) coherence properties are invariant along that direction.

We have also studied a guided partially coherent field propagating in a lossless optical fiber. We have shown that when only the fundamental mode of the fiber is excited, or when all the relevant guided fiber modes are incoherently excited, the field in the fiber is statistically homogeneous along the direction of the axis of the fiber and that its power spectrum and its longitudinal and transverse (second-order) coherence properties do not change as the field propagates along the fiber.

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